

Bipartite graphs with a perfect matching and digraphs *

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Abstract

In this paper, we introduce a corresponding between bipartite graphs with a perfect matching and digraphs, which implicates an equivalent relation between the extendibility of bipartite graphs and the strongly connectivity of digraphs. Such an equivalent relation explains the similar results on k -extendable bipartite graphs and k -strong digraphs. We also study the relation among k -extendable bipartite graphs, k -strong digraphs and combinatorial matrices. For bipartite graphs that are not 1-extendable and digraphs that are not strong, we prove that the elementary components and strong components are counterparts.

Key words: k -extendable, strongly k -connected, indecomposable, irreducible, strong component, elementary component

1 Introduction and terminologies

In this paper, all graphs (digraphs) considered have no loop and multiple edge (arc) unless explicitly stated. For all terminologies not defined, we refer the reader to [2], [3] and [8]. All matrices considered are zero-one matrices.

We use $V(G)$ and $E(G)$ to denote the vertex set and edge set of a graph G . Let G be a bipartite graph with bipartition (U, W) where $U = \{u_1, \dots, u_n\}$ and $W = \{w_1, \dots, w_n\}$. The matrix $A = (a_{ij})_{n \times n}$, where $a_{ij} = 1$ if and only if $u_i w_j \in E(G)$, is called the *reduced adjacency matrix* of G . We denote A by $R(G)$. We call G the *reduced associated bipartite graph* of A and denote G by $B(A)$.

A connected graph is *elementary* if the union of its perfect matchings forms a connected subgraph. A connected graph G is called *k -extendable*, for $k \leq (|V(G)| - 1)/2$, if G has a matching of size k , and every matching of size k of G is contained in a perfect matching of G . G is said to be *minimal k -extendable* if G is k -extendable but $G - e$ is not k -extendable for any $e \in E(G)$. An edge of G is called a *fixed single (fixed double)* edge if it belongs to no (all) perfect matchings of G . An edge of G is called *fixed* if it is either a fixed single or a fixed double edge of G . All non-fixed edges of G form a subgraph H , each component of which is elementary and is therefore called an *elementary component*.

Let D be a digraph. We denote by $V(D)$, $A(D)$ and $M(D)$ the vertex set, arc set and the adjacent matrix of D . Let M be an adjacent matrix of D , we call D the *associated digraph* of M and denote D by $D(M)$. D is *strongly connected*, or *strong*, if there exists a path from x to y and a path from y to x in D for any $x, y \in V(D)$, $x \neq y$. A set $S \subset V(D)$ is a *separator* if $D - S$ is not strong. D is *k -strongly connected*, or *k -strong*, if $|V(D)| \geq k + 1$ and D has no separator of order less than k . D is *minimal k -strong* if D is k -strong, but $D - a$ is not k -strong for any arc $a \in A(D)$. A *strong component* is a maximal subdigraph of D which is strong.

We call a path, directed or undirected, from u to v a (u, v) -path. The set of the end-vertices of the edges in a matching M is denoted by $V(M)$, or $V(e)$ if $M = \{e\}$. The symmetric difference of two sets S_1 and S_2 , is denoted by $S_1 \Delta S_2$.

Let B_n denote the set of all matrices of order n over the Boolean algebra $\{0, 1\}$. We call a matrix $A \in B_n$ *reducible* if there exists a permutation matrix P , such that

$$P^T A P = \begin{bmatrix} B & 0 \\ C & D \end{bmatrix},$$

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where B is an $l \times l$ matrix and D is an $(n-l) \times (n-l)$ matrix, for some $1 \leq l \leq n-1$. A is *irreducible* if it is not reducible. Let k be an integer with $1 \leq k \leq n$. A is called *k-reducible* if there exists a permutation matrix P , such that

$$P^T A P = \begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{21} & A_{22} & A_{23} \end{bmatrix},$$

where A_{11} and $[A_{22} \ A_{23}]$ are square matrices of order at least one and the size of the zero submatrix at the upper right corner is $l \times (n-k+1-l)$, $1 \leq l \leq n-1$. If A is not *k-reducible*, then A is called *k-irreducible*.

A matrix $A \in B_n$ is call *partly decomposable* if there exist permutation matrices P and Q , such that

$$P A Q = \begin{bmatrix} B & 0 \\ C & D \end{bmatrix},$$

where B is an $l \times l$ matrix and D is an $(n-l) \times (n-l)$ matrix, for some $1 \leq l \leq n-1$. A is *fully indecomposable* if it is not partly decomposable. Let k be an integer with $0 \leq k \leq n$. A is called *k-partly decomposable* if it contains an $l \times (n-k+1-l)$ zero submatrix, for some $1 \leq l \leq n-1$. A matrix which is not *k-partly decomposable* is called *k-indecomposable*.

A *diagonal* of a matrix $A = (a_{ij}) \in B_n$ is a collection T of n entries $a_{1i_1}, a_{2i_2}, \dots, a_{ni_n}$ of A such that $\{i_1, i_2, \dots, i_n\} = \{1, 2, \dots, n\}$. If $i_j = j$ for $j = 1, 2, \dots, n$, we call the diagonal *main diagonal* of the matrix.

Let G be a bipartite graph with bipartition (U, W) , where $U = \{u_1, \dots, u_n\}$ and $W = \{w_1, \dots, w_n\}$, and $M = \{u_i w_i, 1 \leq i \leq n\}$ a perfect matching of G . We form $R(G) = (a_{ij})_{n \times n}$, where $a_{ij} = 1$ if and only if $u_i w_j \in E(G)$. Then $R(G)$ has a positive main diagonal, which corresponds to M . We obtain a digraph $D = D(R(G) - I)$, where I denote the identity matrix. On the contrary, given a digraph D , we can get a bipartite graph $G = B(M(D) + I)$, which has a perfect matching. Hence we have a corresponding between bipartite graphs with a perfect matching and digraphs. We may get different D from G , depending on how we choose the perfect matching M , therefore we denote D by $D = D(G, M)$. While G is uniquely determined by D , we denote it by $G = B(D)$. Clearly, such a corresponding includes a bijection between M and $V(D)$, and a bijection between $E(G) \setminus M$ and $A(D)$. D can also be understood as obtained from G by orienting all edges of G towards the same partition and then contracting all edges of M .

There is a well-known equivalent property between the 1-extendibility of G and the strong connectivity of D .

Theorem 1.1. ([8], Exercise 4.1.5) *Let G be a bipartite graph and M a perfect matching of G . Then $D = D(G, M)$ is strong if and only if G is 1-extendable.*

The following is another interesting relation between G and D .

Theorem 1.2. ([8], Exercise 4.3.3) *Let G be a bipartite graph with a unique perfect matching M . Then $D = D(G, M)$ is acyclic.*

In this paper we further discuss the relation between G and D , as well as their relations with combinatorial matrices.

2 Extendibility versus Connectivity

Below is a generalization of Theorem 1.1, which has been stated in [12] without a proof.

Theorem 2.1. *Let G be a bipartite graph and M a perfect matching of G . Then $D = D(G, M)$ is k -strong if and only if G is k -extendable.*

We prove Theorem 2.1 in this section and show some interesting applications of it. We need Menger's Theorem in our proof.

Theorem 2.2. (Menger [10]) *Let D be a digraph. Then D is k -strong if and only if $|V(D)| \geq k+1$ and D contains k internally vertex disjoint (s, t) -paths for every choice of distinct vertices $s, t \in V$.*

Actually we use an equivalent form of Menger's Theorem. Further more, we only need the following weakened form, which appears as an exercise in [2].

Lemma 2.3. ([2], Exercise 7.17) Let D be a k -strong digraph. Let $x_1, x_2, \dots, x_{k-1}, y_1, y_2, \dots, y_{k-1}$ be distinct vertices of D , then there are k independent paths in D , starting at x_i , $0 \leq i \leq k-1$ and ending at y_j , $0 \leq j \leq k-1$.

Now comes the proof of Theorem 2.1.

Proof. Let D be k -strong. We use induction on k to prove that G is k -extendable. When $k = 1$, the conclusion follows from Theorem 1.1. Suppose that the conclusion holds for all integers $1 \leq m < k$. Now we prove that an arbitrary matching M_0 of size k in G is contained in a perfect matching of G .

Firstly we assume that $|M_0 \cap M| \geq 1$. Let $e \in M_0 \cap M$ and the vertex in D corresponding to e be v_e . Let $G' = G - V(e)$, $D' = D - v_e$, and $M' = M \setminus e$. Then D' is $(k-1)$ -strong and $D' = D(G', M')$. By the induction hypothesis, G' is $(k-1)$ -extendable. Hence $M_0 \setminus \{e\}$, which is a matching of size $k-1$ in G' , is contained in a perfect matching M' of G' . Then $M' \cup \{e\}$ is a perfect matching of G containing M_0 .

Now we handle the case that $M_0 \cap M = \emptyset$. In this case, M_0 corresponds to an arc set A_0 of order k of D . The arcs in A_0 form some independent cycles and paths in D . Let the set of cycles formed be $\mathcal{C}_0 = \{C_0, C_1, \dots, C_{s-1}\}$ and the set of paths formed be $\mathcal{P}_0 = \{P_0, P_1, \dots, P_{t-1}\}$. Let the starting and ending vertices of P_i be u_i and v_i , $0 \leq i \leq t-1$. Let V_0 be the union of the set of vertices of cycles in \mathcal{C}_0 and the set of internal vertices of paths in \mathcal{P}_0 . Then $|V_0| = k - t$. By definition, $D - V_0$ is t -strong. By Lemma 2.3, there are t independent paths in D starting at v_i , $0 \leq i \leq t-1$, and ending at u_j , $0 \leq j \leq t-1$. Such paths, together with the paths in \mathcal{P}_0 , form some independent cycles in D . Denote the set of such cycles by \mathcal{C}_1 . Then $\mathcal{C}_0 \cup \mathcal{C}_1$ is a set of independent cycles in D which covers all arcs in A_0 . $\mathcal{C}_0 \cup \mathcal{C}_1$ corresponds to a set \mathcal{C} of independent M -alternating cycles in G . Let the set of edges of cycles in \mathcal{C} be $E(\mathcal{C})$, then $E(\mathcal{C}) \triangle M$ is a perfect matching of G containing M_0 . Hence G is k -extendable.

Conversely, suppose that G is k -extendable. To see that D is k -strong, let $\{v_1, v_2, \dots, v_{k-1}\}$ be a set of $k-1$ vertices in D . Denote by e_i the edge in G corresponds to v_i , $1 \leq i \leq k-1$. Let $G' = G - \bigcup_{i=1}^{k-1} V(e_i)$, $D' = D - \{v_i : 1 \leq i \leq k-1\}$ and $M' = M \setminus \{e_i : 1 \leq i \leq k-1\}$. Then $D' = D(G', M')$. Since G is k -extendable, G' is 1-extendable. Hence D' is strong by Theorem 1.1 and D is k -strong. \square

Theorem 2.4. Let G be a bipartite graph and M a perfect matching of G . If G is minimal k -extendable then $D = D(G, M)$ is minimal k -strong.

Proof. Suppose that G is minimal k -extendable. By Theorem 2.1, D is k -strong. Let a be an arc of D and e be the edge corresponding to a in G . Then $D - a = D(G - e, M)$. By the minimality of G , $G - e$ is not k -extendable, hence $D - a$ is not k -strong by Theorem 2.1. By the arbitrary of a , D is minimal k -strong. \square

The converse of Theorem 2.4 does not generally hold, that is, G does not need to be minimal k -extendable if $D = D(G, M)$ is minimal k -strong. For example, we show a minimal strong digraph D_0 in Figure 1 and $G_0 = B(D_0)$, which is not minimal 1-extendable, in Figure 2.

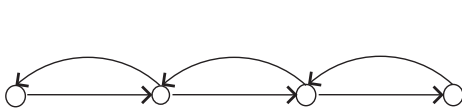


Figure 1: A minimal strong digraph D_0

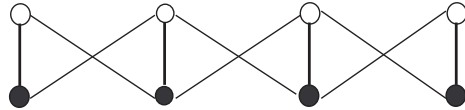


Figure 2: $G_0 = B(D_0)$

There are many parallel results on k -extendable bipartite graphs and k -strong digraphs. Theorem 2.1 and Theorem 2.4 help to explain such a similarity between these two classes of graphs. In the rest of this section, we will illustrate some such results.

Our first demonstrations are the well-known ear decompositions of strong digraphs and 1-extendable bipartite graphs.

An *ear decomposition* of a digraph D is a sequence $\mathcal{E} = \{P_0, P_1, \dots, P_t\}$, where P_0 is a cycle and each P_i is a path, or a cycle with the following properties:

- (a) P_i and P_j are arc disjoint when $i \neq j$.
- (b) For each $i = 1, \dots, t$, if P_i is a cycle, then it has precisely one vertex in common with $V(D_{i-1})$. Otherwise the end-vertices of P_i are distinct vertices of $V(D_{i-1})$ and no other vertex of P_i belongs to

$V(D_{i-1})$. Here D_i denotes the digraph with vertices $\bigcup_{j=0}^i V(P_j)$ and arcs $\bigcup_{j=0}^i A(P_j)$.

(c) $\bigcup_{j=0}^t V(P_j) = V(D)$ and $\bigcup_{j=0}^t A(P_j) = A(D)$.

Theorem 2.5. ([2], Theorem 7.2.2) *A digraph is strong if and only if it has an ear decomposition. Furthermore, if D is strong, then for every vertex v , every cycle C containing v can be used as starting cycle P_0 for an ear decomposition of D .*

Let e be an edge and G_0 be the graph containing e only. Join the end-vertices of e by an odd path P_1 we obtain a graph G_1 . Now if $G_{i-1} = e + P_1 + \dots + P_{i-1}$ has already been constructed, join any two vertices in different color classes of G_{i-1} by an odd path P_i having no other vertices in common with G_{i-1} we obtain G_i . The decomposition $G_r = e + P_1 + \dots + P_r$ is called a *bipartite ear decomposition* of G_r .

Theorem 2.6. ([8], Theorem 4.1.6) *A bipartite graph is 1-extendable if and only if it has a bipartite ear decomposition. Such an ear decomposition may be started with any edge e of G .*

It is remarked in [8] that, given a bipartite ear decomposition $G = e + P_1 + \dots + P_r$ of a bipartite graph G , there is exactly one perfect matching M in G such that $M \cap E(G_i)$ is a perfect matching of G_i for every i , $0 \leq i \leq r$. It is not hard to check that the given bipartite ear decomposition corresponds to an ear decomposition of the digraph $D = D(G, M)$.

Next, we show two corresponding characterizations.

Theorem 2.7. (Plummer [11]) *Let G be a connected bipartite graph with bipartition (U, W) , k a positive integer such that $k \leq (|V(G)| - 2)/2$. Then G is k -extendable if and only if $|U| = |W|$ and for all non-empty subset X of U with $|X| \leq |U| - k$, $|N(X)| \geq |X| + k$.*

Let D be a digraph. Let X, Y be disjoint non-empty proper subsets of $V(D)$, the ordered pair (X, Y) is called a *one-way pair* in D if D has no arc with tail in X and head in Y . Let $h(X, Y) = |V - X - Y|$.

Theorem 2.8. (Frank and Jordán [5]) *A digraph D is k -strong if and only if $h(X, Y) \geq k$ for every one-way pair (X, Y) in D .*

The condition in Theorem 2.8 is equivalent to that $N^+(X) \geq k$ for any set $X \subseteq V(D)$ with $|X| \leq |V(D)| - k$, which is similar to the condition in Theorem 2.7.

The counterpart of Menger's Theorem for bipartite k -extendable graphs was proved by Aldred et al. in [1]. The original proof is a little involved. Now, with Theorem 2.1, we can deduce it from Menger's Theorem straightly.

Theorem 2.9. *Let G be a bipartite graph with bipartition (U, W) and a perfect matching. Then G is k -extendable if and only if for any perfect matching M and for each pair of vertices $u \in U$ and $w \in W$, there are k internally disjoint M -alternating paths connecting u and w , furthermore, these k paths start and end with edges in $E(G) \setminus M$.*

Proof. Let M be any perfect matching of G , and $D = D(G, M)$ be obtained by orienting all edges of G towards W then contracting all edges in M . Suppose that G is k -extendable. Firstly we prove the below claim.

Claim 1. Let D be a k -strong digraph and x a vertex of D , then D contains k cycles, any two of which intersect at x only.

Proof. Let x' be a vertex not in $V(D)$. Construct D' such that $V(D') = V(D) \cup \{x'\}$, $A(D') = A(D) \cup \{ux' : ux \in A(D)\} \cup \{x'u : xu \in A(D)\}$. We prove that D' is k -strong. If D' is not k -strong, then there exists a separator S of size less than k . If S contains x' , then $S - x'$ is a separator of D of size less than $k - 1$, contradicting the strong connectivity of D . Assume that S does not contain x' , then any vertex y which is separated from x' by S is separated from x by S as well, hence S is a separator of D , again contradicting the strong connectivity of D . Therefore D' is k -strong. By Menger's Theorem there are k internally disjoint (x, x') -paths in D . Replacing every arc ux' in these paths with the arc ux , we obtain the cycles as claimed. \square

By Theorem 2.1, D is k -strong. If $uw \notin M$, let $uu', w'w \in M$, and $u_0, w_0 \in V(D)$ be the vertices of D corresponding to edges uu' and $w'w$. By Menger's Theorem there are k internally disjoint paths in D from u_0 to w_0 , which correspond to k M -alternating paths in G from u' to w' , starting and ending

with the edges $u'u$ and ww' , respectively. Furthermore, any two of these M -alternating paths intersect at the edges $u'u$ and ww' only. Removing $u'u$ and ww' from these paths we obtain k internally disjoint M -alternating paths from u to w in G , starting and ending with edges in $E(G) \setminus M$. If $uw \in M$, let $v \in V(D)$ be the vertices of D corresponding to uw . By Claim 1 there are k cycles in D , any two of which intersect at v only. The cycles correspond to k M -alternating cycles in G , any two of which intersect at the edge uw only. Removing uw from the cycles we obtain the paths we want.

Conversely, suppose that for M , any vertices u and w in G , we can always find the M -alternating paths as stated. Let v_1, v_2 be any two vertices in D and u_1w_1, u_2w_2 be the edges in M corresponding to v_1 and v_2 , where $u_i \in U$ and $w_i \in W$, $i = 1, 2$. Then there are k internally disjoint M -alternating paths from u_1 to w_2 , starting and ending with edges in $E(G) \setminus M$. Adding edges u_1w_1 and u_2w_2 to each of the paths, we get k M -alternating paths, corresponding to k internally disjoint paths in D from v_1 to v_2 . Since v_1, v_2 is arbitrarily chosen, by Menger's Theorem, D is k -strong. By Theorem 2.1, G is k -extendable. \square

When considering minimal k -extendable bipartite graph and minimal k -strong digraphs, We find the following similar results.

Theorem 2.10. (Mader [9]) *Every minimal k -strong digraph contains at least k vertices of out-degree k and at least k vertices of in-degree k .*

Theorem 2.11. (Lou [7]) *Every minimal k -extendable bipartite graph G with bipartition (U, W) has at least $2k + 2$ vertices of degree $k + 1$. Furthermore, both U and W contain at least $k + 1$ vertices of degree $k + 1$.*

Neither of them implies the other but striking analogical techniques were used in [9] and [7]. We cite two corresponding structural lemmas here.

Let $h(a)$ and $t(a)$ denote the head and tail of an arc a , respectively. An arc set a_1, a_2, \dots, a_m , where m is even, is called an *anti-directed trail* if for all i , $h(a_{2i+1}) = h(a_{2i+2})$ and $t(a_{2i+2}) = t(a_{2i+3})$, or for all i , $t(a_{2i+1}) = t(a_{2i+2})$ and $h(a_{2i+2}) = h(a_{2i+3})$ (indexes modulo m).

Theorem 2.12. (Mader [9]) *Let D be a minimal k -strong digraph. Then the subgraph of D induced by all arcs whose tail is of outdegree at least $k + 1$ and whose head is of indegree at least $k + 1$ does not contain an anti-directed trail.*

Theorem 2.13. (Lou [7]) *In a minimal k -extendable bipartite graph, the subgraph induced by the edges both ends of which have degree at least $k + 2$ is a forest.*

It can be verified that an anti-directed trail in D corresponds to a closed trail in $G = B(D)$, while a closed trail in G does not always corresponds to an anti-directed trail in D .

3 Combinatorial Matrices

In this section, we show the equivalence among k -connected digraphs, k -extendable bipartite graphs and combinatorial matrices.

Theorem 3.1. ([6], Theorem 2.1.1) *Let $A \in B_n$, then A is irreducible if and only if the associated digraph $D(A)$ is strong.*

Theorem 3.2. (Brualdi et al. [4]) *Let $A \in B_n$. Then A is fully indecomposable if and only if every one entry of A lies in a nonzero diagonal, and every zero entry of A lies in a diagonal with exactly one zero member.*

A nonzero diagonal of A corresponds to a perfect matching of the reduced associated bipartite graph $B(A)$. The condition in Theorem 3.2 is equivalent to that $B(A)$ is 1-extendable.

Theorem 3.3. ([6], Theorem 2.1.3) *Let $A \in B_n$, Then*

- (1) *If A is fully indecomposable, then A is irreducible.*
- (2) *A is irreducible if and only if $A + I$ is fully indecomposable.*

The followings are generalized results for k -indecomposable matrices and k -irreducible matrices.

Theorem 3.4. (You et al. [13]) *Suppose $k \geq 1$. Then a matrix $A \in B_n$ is k -irreducible if and only if $D(A)$ is k -strong.*

Theorem 3.5. Suppose $0 \leq k \leq n-1$ and $A \in B_n$. Then A is k -indecomposable if and only if $G = B(A)$ is k -extendable.

Proof. Suppose that A is k -indecomposable. Let the bipartition of G be (U, W) . Let U_1 be a subset of U such that $|U_1| \leq n-k$. If $|N(U_1)| \leq |U_1| + k - 1$, then $|W \setminus N(U_1)| \geq n - |U_1| - k + 1$, and the submatrix of A indexed by U_1 and $W \setminus N(U_1)$ is a zero matrix of size at least $|U_1| \times (n - k + 1 - |U_1|)$. By definition, A is k -partly decomposable, a contradiction. Hence $|N(U_1)| \geq |U_1| + k$. By Theorem 2.7, G is k -extendable.

Conversely, suppose that G is k -extendable. If A is k -partly decomposable then A has an $l \times (n - k + 1 - l)$ zero submatrix, for some $1 \leq l \leq n - k$. Let the subset of $V(G)$ indexing the row of the submatrix be U_1 , then $|U_1| = l \leq n - k$ and $|N(U_1)| \leq n - (n - k + 1 - l) = l + k - 1 = |U_1| + k - 1$, contradicting Theorem 2.7. \square

Lemma 3.6. (You et al. [13]) Suppose $k \geq 1$ and $A \in B_n$ has a positive main diagonal. Then A is k -indecomposable if and only if $D(A)$ is k -strong.

Theorem 3.7. Let $A \in B_n$, Then

- (1) If A is k -indecomposable, then A is k -irreducible.
- (2) A is k -irreducible if and only if $A + I$ is k -indecomposable.

Proof. By definition, if A is k -reducible then A is k -decomposable. Hence if A is k -indecomposable, A is k -irreducible and (1) holds.

By Theorem 3.4, A is irreducible if and only if $D(A)$ is k -strong. Since adding a loop to a vertex or removing a loop from a vertex does not affect the strongly connectivity of a digraph, $D(A)$ is k -strong if and only if $D(A + I)$ is k -strong. By Lemma 3.6, $D(A + I)$ is k -strong if and only if $A + I$ is k -indecomposable. \square

4 Elementary components versus strong components

Let G be a bipartite graph with a perfect matching M , but not 1-extendable. By Theorem 2.1, $D = D(G, M)$ is not strong. In this section, we consider the elementary components of G and the strong components of D .

Lemma 4.1. Let G be a bipartite graph with a perfect matching M , and G_1 an elementary component of G , then $E(G_1) \cap M$ is a perfect matching of G_1 .

Proof. An edge $e \in E(G) \setminus E(G_1)$ incident to a vertex in G_1 is fixed. However it can not be a fixed double edge, since every edge adjacent to a fixed double edge must be a fixed single edge. Hence, all edges in M saturating vertices in $V(G_1)$ must be in $E(G_1)$ and $E(G_1) \cap M$ is a perfect matching of G_1 . \square

Let $M_1 = E(G_1) \cap M$, then $D_1 = D(G_1, M_1)$ is a subdigraph of D . Moreover, let G_1 be a subgraph of G consisting of only a fixed double e edge of G , then $e \in M$ and $D_1 = D(G_1, \{e\})$ contains only one vertex of D .

Theorem 4.2. Let G be a bipartite graph with a perfect matching M , G_1 a subgraph of G such that $M_1 = E(G_1) \cap M$ is a perfect matching of G_1 . Let $D = D(G, M)$ and $D_1 = D(G_1, M_1)$. Then the followings are equivalent.

- (1) G_1 is an elementary component of G , or consists of a fixed double edge only.
- (2) D_1 is a strong component of D .

Proof. Suppose that G_1 is an elementary component of G . Then G_1 is 1-extendable and hence D_1 is strong. Assume that D_1 is properly contained in a strong subdigraph D'_1 of D . Then $G'_1 = B(D'_1)$ is a 1-extendable subgraph of G containing G_1 . Furthermore, any perfect matching of G'_1 is contained in a perfect matching of G . Therefore any edge of G'_1 is contained in a perfect matching of G . However any edge in $E(G'_1) \setminus E(G_1)$ incident to a vertex of G_1 must be a fixed single edge and can not be contained in any perfect matching of G , which leads to a contradiction. Hence D_1 is a maximal strong subdigraph, that is, a strong component, of D .

Suppose that G_1 consists of a fixed double edge e only. Then $e \in M$ and $D(G_1, \{e\})$ contains exactly a vertex v in D . If v is properly contained in a strong component D'_1 of D , then $G'_1 = B(D'_1)$ is a 1-extendable subgraph of G containing e . Furthermore, every perfect matching of G'_1 is contained in a perfect matching of G . Hence every edge of G'_1 is contained in a perfect matching of G . However e is contained in every perfect matching of G , so all edges adjacent to e are fixed single edges and cannot be

contained in a perfect matching of G , a contradiction. Hence v composes a strong component of D with only one vertex.

Conversely, let D_1 be a strong component of D . Then $G_1 = B(D_1)$ is 1-extendable. To prove that G_1 is an elementary component or consist of a fixed double edge, we need only to prove that an edge $e = u_1u_2 \in E(G) \setminus E(G_1)$ associated with a vertex $u_1 \in V(G_1)$ is a fixed single edge. Suppose that e is not a fixed single edge and contained in a perfect matching M' of G . Let $u_1w_1, u_2w_2 \in M$, which correspond to vertices v_1 and v_2 in D respectively, then $v_1 \in V(D_1)$ and $v_2 \notin V(D_1)$. $M \triangle M'$ consists of nonadjacent edges and alternating cycles. The edges e, u_1w_1 and u_2w_2 must be contained in an alternating cycle C . However C corresponds to a directed cycle in D , which contains v_1 and v_2 . This contradicts the fact that D_1 is a strong component of D . \square

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